UK INTERMEDIATE MATHEMATICAL CHALLENGE

February 6th 2014

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SOLUTIONS AND INVESTIGATIONS

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some suggestions for further investigations.

The Intermediate Mathematical Challenge (IMC) is a multiple choice contest, in which you are presented with five alternative solutions, of which just one is correct. It follows that often you can find the correct solutions by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the IMC, and we often give first a solution using this approach.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. So usually we have included a complete solution which does not use the fact that one of the given alternatives is correct. Thus we have aimed to give full solutions with all steps explained. We therefore hope that these solutions can be used as a model for the type of written solution that is expected in the Intermediate Mathematical Olympiad and similar competitions.

We welcome comments on these solutions, and, especially, corrections or suggestions for improving them. Please send your comments,

either by e-mail to enquiry@ukmt.org.uk

or by post to IMC Solutions, UKMT Maths Challenges Office, School of Mathematics, University of Leeds, Leeds LS2 9JT.

Quick Marking Guide

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
A	**D**	**E**	**C**	**E**	**E**	**B**	**C**	**A**	**B**	**C**	**D**	**B**	**B**	**E**	**C**	**D**	**B**	**C**	**D**	**D**	**B**	**A**	**A**	**E**

1. What is 25% of $\frac{3}{4}$?

A $\dfrac{3}{16}$ B $\dfrac{1}{4}$ C $\dfrac{1}{3}$ D 1 E 3

Solution: **A**

As 25% of a number is one quarter of it, 25% of $\frac{3}{4}$ is $\frac{1}{4} \times \frac{3}{4} = \frac{3}{16}$.

For further investigation

1.1 What percentage of $\frac{3}{4}$ is $\frac{2}{3}$?

2. Which is the smallest positive integer for which all these are true?
(i) It is odd.
(ii) It is not prime.
(iii) The next largest odd integer is not prime.

A 9 B 15 C 21 D 25 E 33

Solution: **D**

In the context of the IMC we can find the solution by eliminating all but one of the options. We first note that all the options are odd numbers that are not primes. So conditions (i) and (ii) are true for all of them. Next we see that $9+2$, $15+2$ and $21+2$ are all primes, so for the first three options condition (iii) is not true. However $25+2=27$ is not prime, so all three conditions are satisfied by option D. The same is the case for option E, but, as $25<33$, 25 is the smaller of the given options for which all three conditions are true.

If we were not given the options, we would have to search the sequence of odd numbers to find the first pair of consecutive odd integers that are not primes. The first few odd numbers are

1, 3, 5, 7, **9**, 11, 13, **15**, 17, 19, **21**, 23, **25**, **27**, 29, 31, **33**, **35**, …

where the non-primes are shown in bold. [Don't forget that the number 1 is not a prime number.] From this list we see that 25 is the smallest positive integer for which all three conditions are true.

For further investigation

2.1 Find the next pair of consecutive odd numbers after the pair 33, 35 that are both not primes.

2.2 Find the first triple of consecutive odd numbers which are all not primes.

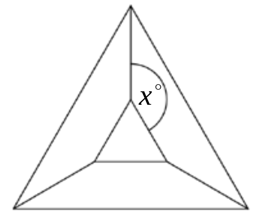
2.3 Is it true that for each integer $k \geq 2$, there are k consecutive odd numbers that are not primes?

2.4 In fact, it is not difficult to show that for each positive integer n there are n consecutive integers none of which is a prime. Can you find a proof of this? It follows from this that the answer to 2.3 is "yes".

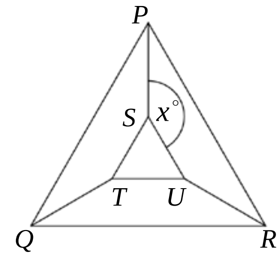
3. An equilateral triangle is placed inside a larger equilateral triangle so that the diagram has three lines of symmetry.

What is the value of x?

A 100 B 110 C 120 D 130 E 150



Solution: **E**



We label the vertices in the diagram as shown.

Because the figure is symmetrical about the line *PS*, $\angle PSU = \angle PST$.

Since the triangle *STU* is equilateral, $\angle TSU = 60^\circ$.

The angles at a point have sum 360°. Therefore

$$\angle PSU + \angle PST + \angle TSU = 360^\circ$$

and so

$$\angle PSU + \angle PSU + 60^\circ = 360^\circ.$$

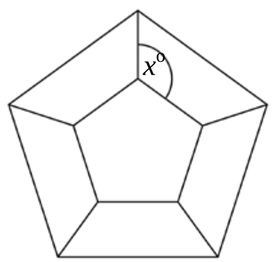
It follows that

$$\angle PSU = \tfrac{1}{2}(360^\circ - 60^\circ) = \tfrac{1}{2}(300^\circ) = 150^\circ.$$

Therefore $x = 150$.

For further investigation

3.1 Solve the analogous problem for the case where the equilateral triangles are replaced by regular pentagons, as shown in the diagram below.



3.2 Find a formula for x for the analogous problem where the equilateral triangles are replaced by regular polygons with n sides, where n is an integer with $n \geq 3$. Check that your formula does give $x = 150$ when $n = 3$, and that it agrees with your answer to 3.1.

3.3 What is the value of n if $x = 105$?

4. You are given that m is an even integer and n is an odd integer. Which of these is an odd integer?

A $3m+4n$ B $5mn$ C $(m+3n)^2$ D m^3n^3 E $5m+6n$

Solution: **C**

We make repeated use of the standard facts that the sum of two even integers is even, the sum of two odd integers is also even, and the sum of an even integer and an odd integer is odd. Also, the product of two integers is odd if, and only if, both integers are odd. Note that the following tables give us a convenient way to summarize these facts.

+	*even*	*odd*
even	*even*	*odd*
odd	*odd*	*even*

×	*even*	*odd*
even	*even*	*even*
odd	*even*	*odd*

We can now check the options, one by one.

A: As m is even, $3m$ is even. As 4 is even, $4n$ is even. So $3m+4n$ is even.

B: As m is even, mn is even. So $5mn$ is even.

C: As 3 and n are both odd, $3n$ is odd. Therefore, as m is even, $m+3n$ is odd. So $(m+3n)^2$ is odd.

In the context of the IMC we could stop here, as we are entitled to assume that there is just one correct answer amongst the given options. For a complete solution, we would need to check that options D and E are also even. We leave this to the reader.

Note that there is a very quick method here which depends on the assumption that whether a particular option is odd or even depends only on whether m and n are odd or even, and not on their actual values. Granted this assumption, we can check the options by substituting any even number for m and any odd number for n. The arithmetic is easiest if we make the choices $m=0$ and $n=1$. Then the values of the options are 4, 0, 9, 0 and 6 respectively. It is then easy to see that only 9, corresponding to option C, is odd.

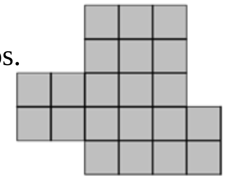
5. A ship's bell is struck every half hour, starting with one bell at 0030, two bells (meaning the bell is struck twice) at 0100, three bells at 0130 until the cycle is complete with eight bells at 0400. The cycle then starts again with one bell at 0430, two bells at 0500 and so on. What is the total number of times the bell is struck between 0015 on one day and 0015 on the following day?

A 24 B 48 C 108 D 144 E 216

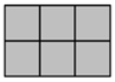
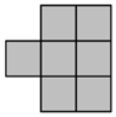
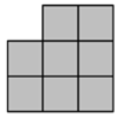
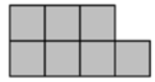
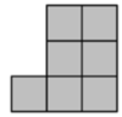
Solution: **E**

In a single cycle of 4 hours the bell is struck $1+2+3+4+5+6+7+8=36$ times. In a 24 hour day there are 6 of these cycles. So in the 24 hours between 0015 one day and 0015 the next day the bell is struck $6\times 36=216$ times.

6. The shape shown on the right was assembled from three identical copies of one of the smaller shapes below, without gaps or overlaps.

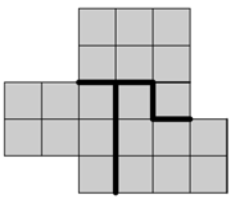


Which smaller shape was used?

A  B  C  D  E 

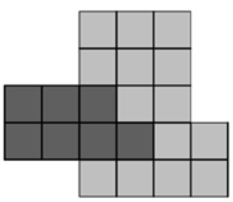
Solution: **E**

The large shape is made up of 21 small squares and so can only be assembled by using three copies of a shape made up of 7 squares. This rules out options A and C. The diagram on the left below shows how the large shape may be assembled from three copies of shape E.



In the IMC we are entitled to assume that only one of the given options is correct. So we could stop here.

However, it is worth considering how we might show convincingly that neither option B nor D can be used. It is easier to think of trying to fit three copies of these shapes into the larger shape without gaps or overlaps, than of assembling the larger shape out of them. If we concentrate on the 2×2 block of squares on the left of the larger shape, we can easily see that there is no way that copies of the shape of option B can be fitted into the larger shape to cover these 4 squares. The only way that the shape of option D can do this is shown below. It is easy to see that it is not possible to fit two more copies of the shape into the remaining squares without overlaps.



So neither option B nor option D will work. We therefore conclude that the shape of option E is the only one that we can use to make the larger shape.

7. Just one positive integer has exactly 8 factors including 6 and 15.

What is the integer?

A 21 B 30 C 45 D 60 E 90

Solution: **B**

We can rule out options A and C as neither 21 nor 45 has 6 as a factor. We can now complete the solution by looking at the different factors of the other three options.

We see that the factors of 30 are: 1, 2, 3, 5, 6, 10, 15, 30. So 30 has exactly 8 factors and it is divisible by 6 and 15. So, in the context of the IMC, we could stop here.

However this approach does not prove that 30 is the *only* positive integer divisible by 6 and 15 which has exactly 8 factors. So we now ask you to investigate this problem in more depth.

For further investigation

You should know that each positive integer, other than 1, can be factorized into prime numbers in just one way. By this we mean that, although we may order the prime factors of a number in different ways, the primes that occur, and the number of times they occur, are uniquely determined by the number we start with. This fact is often called the *Fundamental Theorem of Arithmetic.*

For example we can express 90 as a product of primes in more than one order, such as $2\times3\times5\times3$ or $5\times3\times3\times2$, but the prime factors will always be one 2, two 3s and one 5. Using index notation we can write $90=2^1 3^2 5^1$. This is the standard way of writing the prime factorization of a positive integer, with the primes in increasing order.

7.1 Give the prime factorizations of 21, 30, 45, 60 and 90.

Once we have found the prime factorization of a positive integer it is easy to work out how many different factors it has. Here is the formula:

> If the positive integer *n* has the prime factorization $p^a q^b r^c ....$ then it has
> $$(a+1)(b+1)(c+1)...$$
> different factors.

For example, 60 has the prime factorization $2^2 3^1 5^1$. The formula above then tells us that 60 has $(2+1)(1+1)(1+1)=3\times2\times2=12$ different factors.

7.2 List the factors of 60 and check that it has exactly 12 factors.

7.3 List the factors of 45 and 120. Check that the above formula gives the correct number of factors in both these cases.

Although we wrote the general prime factorization of *n* as $p^a q^b r^c ...$, our formula also covers the case of positive integers with just 1 or 2 different prime factors. That is, a positive integer with the prime factorization p^a has $(a+1)$ different factors, and one with prime factorization $p^a q^b$ has $(a+1)(b+1)$ factors. For example $32=2^5$ and so has $(5+1)=6$ different factors, and $21=3^1 7^1$ and so has $(1+1)(1+1)=4$ factors.

7.4 Check this is correct by listing all the factors of 32 and of 21.

7.5 If a number is divisible by 6 and by 15 which prime factors must it have?

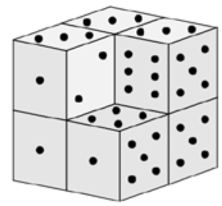
7.6 What possibilities are there for the prime factorization of an integer which is divisible by 6 and by 15 and which has exactly 8 factors? Deduce that 30 is the only positive integer that has exactly 8 factors including 6 and 15.

7.7 Explain why the formula $(a+1)(b+1)(c+1)...$ for the number of factors of a positive integer with prime factorization $p^a q^b r^c ...$ is correct.

7.8 Which positive integers have an odd number of factors?

7.9 Which is the smallest positive integer which has exactly 15 factors?

8. A large cube is made by stacking eight dice. The diagram shows the result, except that one of the dice is missing. Each die has faces with 1, 2, 3, 4, 5 and 6 pips and the total number of pips on opposite faces is 7. When two dice are placed face to face, the matching faces must have the same number of pips.

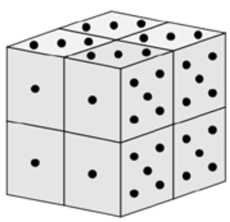


What could the missing die look like?

A  B  C  D  E 

Solution: **C**

Although all standard dice have pips adding up to 7 on opposite faces, dice can vary according to their orientation. From the diagram showing the completed cube on the left below, we see



that, for the missing die, the faces with 1, 3 and 5 pips go round the vertex where these faces meet in clockwise order. [In a differently oriented die, these faces would go round anti-clockwise. If you have a die to hand, check how these faces are oriented on it.]

So we need to decide which of the dice given as options have this orientation of these three faces. In the die of option A, the face with 3 pips is on the bottom, and we can visualize that the 1, 3 and 5 pip faces go round in anti-clockwise order. In the die of option B, the face with 5 pips is the opposite face to that with 2 pips, and we can again visualize that the 1, 3 and 5 pip faces go round anticlockwise. In the die of option C the face with 3 pips is opposite the face with 4 pips, and the face with 5 pips is opposite that with 2 pips. You should be able to visualize that this means that the faces with 1, 3 and 5 go round in clockwise order. So C is the correct option.

Of course, to complete the question, we really need to check that in the dice of options D and E the 1, 3 and 5 pip faces go round in anti-clockwise order. This is left to you to do.

9. At the age of twenty-six, Gill has passed her driving test and bought a car. Her car uses p litres of petrol per 100 km travelled. How many litres of petrol would be required for a journey of d km?

A $\frac{pd}{100}$ B $\frac{100p}{d}$ C $\frac{100d}{p}$ D $\frac{100}{pd}$ E $\frac{p}{100d}$

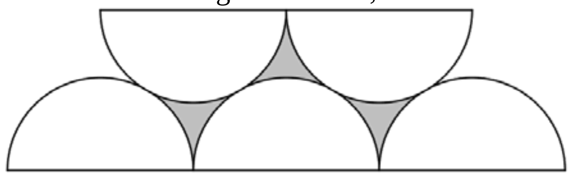
Solution: **A**

The car consumes p litres of petrol when travelling 100 km, and so uses $\frac{p}{100}$ litres for each km.

Therefore in travelling d km the number of litres of petrol used is $\frac{p}{100} \times d = \frac{pd}{100}$.

Note that there is an easy way to check this answer. We know the car uses p litres when it travels 100 km, so the correct formula must have value p when $d = 100$. So only options A and B could possibly be correct. However, the formula of option B implies that the larger d is, that is, the further you travel, the less petrol you use. This cannot be right. This leaves A as the only option which could be correct.

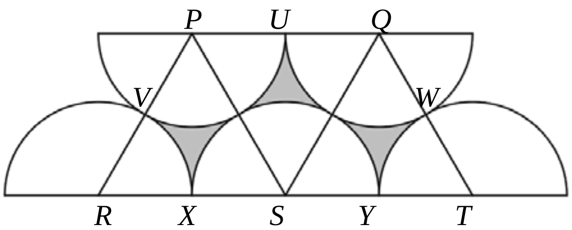
10.	The diagram shows five touching semicircles, each with radius 2.



What is the length of the perimeter of the shaded shape?

A 5π B 6π C 7π D 8π E 9π

Solution: **B**



In the diagram on the left, the centres *P*, *Q*, *R*, *S* and *T* of the semicircles, and some of points where they touch have been labelled. Since the semicircles with centres *P* and *R* touch at *V* , the line joining *P* and *R* goes through *V*. Each semicircle has radius 2, and it follows that *PR* has length 4. Similarly, each of the line segments *PQ*, *PS*, *QS*, *QT*, *RS* and *ST* has length 4. It follows that each of the triangles *PRS*, *PSQ* and *QST* is equilateral and hence each of the angles in them is 60°.

Therefore $\angle VPU = 120^\circ$ and $\angle WQU = 120^\circ$. So the parts of the semicircles with centres *P* and *Q* which make up part of the perimeter of the shaded shape each consists of two thirds of the relevant semicircle. Similarly, as $\angle VRX = 60^\circ$ and $\angle WTY = 60^\circ$, the parts of the perimeter which form parts of the semicircles with centres *R* and *S* consist of one third of these semicircles. Also, the whole of the semicircle with centre *S* is part of the perimeter of the shaded region. It follows that the length of the perimeter is the same as that of $2\times\frac{2}{3}+2\times\frac{1}{3}+1=3$ semicircles with radius 2. So the length of the perimeter of the shaded region is $3\times 2\pi = 6\pi$.

For further investigation

10.1 What is the area of the shaded shape in the diagram of Question 10?

11.	Not all characters in the Woodentops series tell the truth. When Mr Plod asked them, “How many people are there in the Woodentops family?”, four of them replied as follows:

Jenny: “An even number.” Willie: “An odd number.” Sam: “A prime number.”

Mrs Scrubitt: “A number which is the product of two integers greater than one.”

How many of these four were telling the truth?

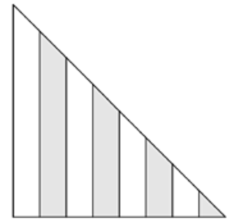
A 0 B 1 C 2 D 3 E 4

Solution: **C**

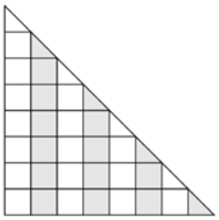
The number of people in the Woodentops family is a positive integer which is greater than one. Every such integer is either even or odd but not both. So precisely one of Jenny and Willie is telling the truth, but we don’t know which.

Also, every integer greater than one is either a prime number or the product of two integers greater than one, but not both. So precisely one of Sam and Mrs Scrubitt is telling the truth, but, again, we don’t know which.

It follows that exactly two of them are telling the truth, though we don’t know which two.

12. The diagram shows an isosceles right-angled triangle divided into strips of equal width. Four of the strips are shaded. What fraction of the area of the triangle is shaded? A $\frac{11}{32}$ B $\frac{3}{8}$ C $\frac{13}{32}$ D $\frac{7}{16}$ E $\frac{15}{32}$	

Solution: **D**



Method 1

If the triangle is divided up as shown on the left we see that it is divided into 28 equal squares and 8 half squares, of which 12 squares and 4 half squares are shaded. Hence the fraction of the area of the triangle that is shaded is

$$\frac{12+4\times\frac{1}{2}}{28+8\times\frac{1}{2}}=\frac{14}{32}=\frac{7}{16}.$$

Method 2

We can choose units so that the two equal sides of the triangle have length 8. So the area of the triangle is $\frac{1}{2}(8\times 8)=32$.

Then the shaded half square at the right of the diagram has area $\frac{1}{2}$ and we see that as we move to the left each shaded strip has area 2 greater than the previous shaded strip. So the total area of the shaded strips is $\frac{1}{2}+2\frac{1}{2}+4\frac{1}{2}+6\frac{1}{2}=14$. So the fraction of the area of the triangle that is shaded is $\frac{14}{32}=\frac{7}{16}$.

For further investigation

12.1 What fraction of the area of the triangle would be shaded if it were divided into 10 strips of equal width of which five alternate strips, starting with the smallest, are shaded?

12.2 What fraction of the area of the triangle would be shaded if it were divided into $2n$ strips of equal width of which n alternate strips, starting with the smallest, are shaded?

12.3 What can you say about your answer to 12.2 as the integer n gets larger and larger?

13. How many numbers can be written as a sum of two different positive integers each at most 100? A 100 B 197 C 198 D 199 E 200

Solution: **B**

The only numbers than can be written as the sum of two positive integers are positive integers. The smallest positive integer that can be written in the required form is $3=1+2$ and the largest is $199=99+100$. Every other positive integer between 3 and 199 can also be written as the sum of two different positive integers at each most 100, as the integers from 4 to 101 may be written as $1+k$ with $3\le k\le 100$, and the integers from 102 to 198 may be written as $k+100$ with $2\le k\le 98$.

There are 197 integers from 3 to 199 inclusive, so this is the number of integers that may be written in the required form.

For further investigation

13.1 How many numbers can be written as the sum of two different positive integers each less than the positive integer n?

14. This year the *Tour de France* starts in Leeds on 5 July. Last year, the total length of the *Tour* was 3404 km and the winner, Chris Froome, took a total time of 83 hours 56 minutes 40 seconds to cover this distance. Which of these is closest to his average speed over the whole event?

A 32km/h B 40 km/h C 48 km/h D 56 km/h E 64 km/h

Solution: **B**

As 3404 is close to 3400 and 83 hours 56 minutes 40 seconds is almost 84 hours, his average speed is close to $\frac{3400}{84}$ km/h. Now $\frac{3400}{84} = \frac{850}{21} = 40\frac{10}{21}$. So, of the given options, 40 km/h is the closest to his average speed.

15. Zac halves a certain number and then adds 8 to the result. He finds that he obtains the same answer if he doubles his original number and then subtracts 8 from the result.

What is Zac's original number?

A $8\frac{2}{3}$ B $9\frac{1}{3}$ C $9\frac{2}{3}$ D $10\frac{1}{3}$ E $10\frac{2}{3}$

Solution: **E**

Let Zac's original number be *x*. When Zac halves this and adds 8 he obtains $\frac{1}{2}x + 8$. When he doubles his original number and subtracts 8, the result is $2x - 8$. As these answers are the same $\frac{1}{2}x + 8 = 2x - 8$. Therefore $\frac{3}{2}x = 16$. Hence $x = \frac{2}{3} \times 16 = \frac{32}{3} = 10\frac{2}{3}$.

16. The base of a triangle is increased by 25% but the area of the triangle is unchanged.

By what percentage is the corresponding perpendicular height decreased?

A $12\frac{1}{2}$% B 16% C 20% D 25% E 50%

Solution: **C**

When the base is increased by 25%, it is multiplied by $\frac{5}{4}$. Since the area of the triangle, which is half the base multiplied by the height, remains unchanged, to compensate, the perpendicular height must have been multiplied by $\frac{4}{5}$. Now, $\frac{4}{5} = \frac{80}{100}$. So the new height is 80% of the original height and so the perpendicular height has been decreased by 20%.

17.	How many weeks are there in $8\times7\times6\times5\times4\times3\times2\times1$ minutes?				
	A 1	B 2	C 3	D 4	E 5

Solution: **D**

There are $60=6\times5\times2$ minutes in an hour, $24=8\times3$ hours in a day and 7 days in a week. It follows that the number of weeks in $8\times7\times6\times5\times4\times3\times2\times1$ minutes is

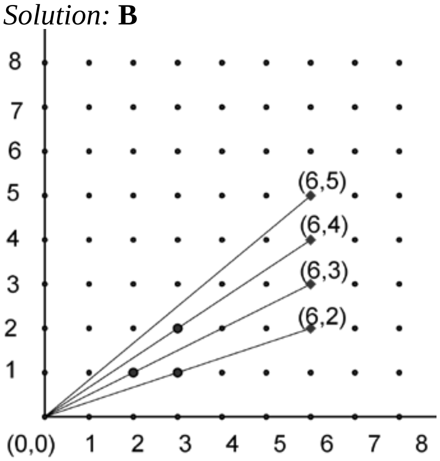
$$\frac{8\times7\times6\times5\times4\times3\times2\times1}{(6\times5\times2)\times(8\times3)\times7}=4\times1=4\text{, after cancelling.}$$

18. Consider looking from the origin (0,0) towards all the points (m,n), where each of m and n is an integer. Some points are *hidden*, because they are directly in line with another nearer point. For example, (2,2) is hidden by (1,1).

How many of the points (6,2), (6,3), (6,4) and (6,5) are *not* hidden points?

A 0 B 1 C 2 D 3 E 4

Solution: **B**



We can see that

the point (6,2) is hidden behind (3,1),

the point (6,3) is hidden behind (2,1),

and the point (6,4) is hidden behind (3,2),

but the point (6,5) is not hidden behind any other point.

So just 1 of the four given points is not hidden.

For further investigation

18.1 Which of the points (6,18), (6,19), (6,20), (6,21), (6,22), (6,23), (6,24), (6,25) are hidden?

18.2 Find a general condition on the integers m and n for the point (m,n) to be hidden.

19.	Suppose that $8^m=27$. What is the value of 4^m?				
	A 3	B 4	C 9	D 13.5	E there is no such m

Solution: **C**

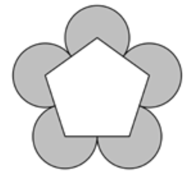
Notice that $8=2^3$ and $27=3^3$. So $8^m=27$ may be written as $(2^3)^m=3^3$. So $2^{3m}=3^3$, and hence $(2^m)^3=3^3$. It follows that $2^m=3$. So $4^m=(2^2)^m=2^{2m}=(2^m)^2=3^2=9$.

For further investigation

19.1 Find the solution of the equation $8^m=27$.

19.2 Suppose that $27^n=8$. What is the value of 81^n?

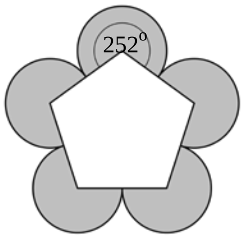
20. The diagram shows a regular pentagon and five circular arcs. The sides of the pentagon have length 4. The centre of each arc is a vertex of the pentagon, and the ends of the arc are the midpoints of the two adjacent edges.



What is the total shaded area?

A 8π B 10π C 12π D 14π E 16π

Solution: **D**



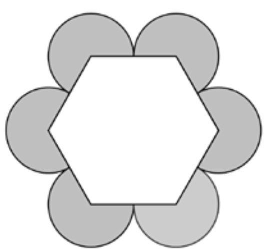
The shaded area is made up of 5 sectors of circles each of whose radius is half the side length of the pentagon, that is, 2. From the formula πr^2 for the area of a circle with radius 2, we see that each of the complete circles has area 4π.

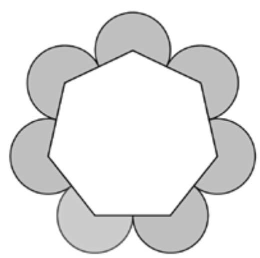
The interior angle of a regular pentagon is 108°. Therefore each of the sectors of the circles subtends an angle $360^\circ - 108^\circ = 252^\circ$ at the centre of the circle. So the area of each sector is $\frac{252}{360} = \frac{7}{10}$ of the area of the circle. It follows that the total shaded area is $5 \times \left(\frac{7}{10} \times 4\pi\right) = 14\pi$.

For further investigation

20.1 The above solution uses the fact that the interior angle of a regular pentagon is 108°. Prove that this is correct.

20.2 Find the total shaded area for the analogous diagrams in which the pentagon is replaced by (a) a regular hexagon with side length 4, and (b) a regular heptagon with side length 4, as shown in the diagrams below.





20.3 Find a formula which gives the total shaded area for the analogous diagram in which the pentagon is replaced by a regular polygon with n sides each of length 4, where n is a positive integer with $n \geq 3$.

21.	In King Arthur's jousting tournament, each of the several competing knights receives 17 points for every bout he enters. The winner of each bout receives an extra 3 points. At the end of the tournament, the Black Knight has exactly one more point than the Red Knight. What is the smallest number of bouts that the Black Knight could have entered? A 3 B 4 C 5 D 6 E 7

Solution: **D**

Suppose the Black Knight enters q bouts and wins x of them and the Red Knight enters r bouts and wins y of them. Then the Black Knight receives $17q + 3x$ points and the Red Knight receives $17r + 3y$ points. As the Black Knight has exactly one more point than the Red Knight

$$(17q + 3x) - (17r + 3y) = 1$$

that is,

$$17(q - r) + 3(x - y) = 1 \qquad (1)$$

The conditions of the problem also imply that

q, r, x and y are integers that satisfy $0 \le x \le q$ and $0 \le y \le r$. (2)

We seek the solution of (1) subject to (2) in which q takes the smallest possible value.

If we put $m = q - r$ and $n = x - y$, the equation (1) becomes

$$17m + 3n = 1 \qquad (3)$$

It is easy to spot that $m = -1$, $n = 6$ is one solution of (3). In this case $x - y = 6$, so $x = y + 6$. As $0 \le y$, the least possible value of x is 6 when $y = 0$. Hence the least possible value of q is 6 and as $q - r = -1$, in this case $r = 7$. So there is a solution in which the Red Knight enters 7 bouts losing them all, and the Black Knight enters 6 bouts winning them all.

It is clear that in any another solution of (3) with $m < 0$, we have $m \le -2$ and hence $n > 6$. So $x = y + n > 6$, and hence, as $q \ge x$, we have $q > 6$ and so we do not obtain a solution with a smaller value of q in this way.

The solution of (3) in which m takes its smallest possible positive value is given by $m = 2$, $n = -11$. In this case $q - r = 2$ and $x - y = -11$. Therefore $y = x + 11$ and hence $y \ge 11$ and so $r \ge 11$. Then $q = r + 2 \ge 13$. So this does not lead to a solution with $q < 6$.

It is clear that in any other solution of (3) with $n < -11$, we must have $r > 11$ and hence $q \ge r > 11$.

We therefore can deduce that the smallest number of bouts that the Black Knight could have entered is 6.

For further investigation

21.1 The equation $17m + 3n = 1$ is called a *linear Diophantine equation.* Diophantus of Alexandria was a Greek mathematician who wrote about finding integer solutions of equations in his influential book *Arithmetica.* He is sometimes called the father of algebra, but little is known about his life. The method for solving linear Diophantine equations is closely connected with the *Euclidean algorithm* for finding the greatest common divisor (highest common factor) of two numbers. Use the web to see what you can find out about Diophantus, the Euclidean algorithm and the solution of linear diophantine equations.

22.	The positive integers *a*, *b* and *c* are all different. None of them is a square but all the products *ab*, *ac* and *bc* are squares. What is the least value that $a+b+c$ can take?				
	A 14	B 28	C 42	D 56	E 70

Solution: **B**

We suppose that r^2, s^2 and t^2 are the largest squares that are factors of *a*, *b* and *c*, respectively. Then there are positive integers *u*, *v* and *w* such that $a = ur^2$, $b = vs^2$ and $c = wt^2$. As r^2 is the largest square that is a factor of *a*, *u* is not divisible by any squares, other than 1, and as *a* is not a square, $u \neq 1$. So *u* is either a prime number or a product of distinct prime numbers. By the same reasoning this is also true of *v* and *w*. Now $ac = uvr^2s^2$ is a square. It follows that *uv* must be a square, and hence, as *u* and *v* are each either prime or the product of distinct primes, they have exactly the same prime factors. Therefore $u = v$. Similarly $v = w$, so $u = v = w$. We have already noted that $u \neq 1$.

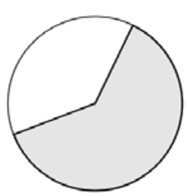
So $a = ur^2$, $b = us^2$ and $c = ut^2$. Since *a*, *b* and *c* are all different, r^2, s^2 and t^2 are all different.

Now, $a+b+c = u(r^2+s^2+t^2)$. So the least value of $a+b+c$ is obtained by choosing *u* to be as small as possible, and $r^2+s^2+t^2$ to be as small as possible. As $u \neq 1$ this means we need to take $u = 2$, and as r^2, s^2 and t^2 are all different, they must be the three smallest squares. This means that the least possible value of $a+b+c$ is $2(1^2+2^2+3^2) = 2(1+4+9) = 2\times 14 = 28$.

Note that it follows that *a*, *b* and *c* are 2, 8 and 18, in some order.

23.	A sector of a disc is removed by making two straight cuts from the circumference to the centre. The perimeter of the sector has the same length as the circumference of the original disc. What fraction of the area of the disc is removed?				
	A $\dfrac{\pi-1}{\pi}$	B $\dfrac{1}{\pi}$	C $\dfrac{\pi}{360}$	D $\dfrac{1}{3}$	E $\dfrac{1}{2}$

Solution: **A**



Suppose that the disc has radius *r*, so that its circumference is $2\pi r$. The ratio of the area of the circle to that of the sector is the same as the ratio of the circumference of the circle to the length of the arc which is part of the perimeter of the sector. Let this common ratio be 1:*k*. Then the arc has length $2\pi rk$. The perimeter of the sector is made up of this arc and two radii of the circle and so its length is $2\pi rk + 2r$. Since this is equal to the circumference of the circle, we have

$$2\pi rk + 2r = 2\pi r.$$

Therefore

$$2\pi rk = 2\pi r - 2r$$

and hence

$$k = \frac{2\pi r - 2r}{2\pi r} = \frac{\pi - 1}{\pi}.$$

24. How many 4-digit integers (from 1000 to 9999) have at least one digit repeated?

A 62×72 B 52×72 C 52×82 D 42×82 E 42×92

Solution: **A**

There are 9000 integers from 1000 to 9999. Instead of counting those with at least one digit repeated, it is easier to count the number with all different digits, and then subtract this total from 9000.

Consider an integer in this range all of whose digits are different. Its first (thousands) digit may be chosen from any of the 9 different non-zero digits. Thereafter we have all 10 digits to choose from other than those that have already been used. So the second (hundreds) digit may be chosen in 9 ways, the third (tens) in 8 ways and the last (units) in 7 ways. So there are $9 \times 9 \times 8 \times 7$ four-digit numbers with no digit repeated.

Therefore the number of 4-digit numbers which have at least one digit repeated is

$$9000 - 9 \times 9 \times 8 \times 7 .$$

It remains only to show that this can be written as one of the five products given as options.

Since $1000 = 8 \times 125$, we have $9000 = 9 \times 8 \times 125$ and therefore

$$\begin{aligned} 9000 - 9 \times 9 \times 8 \times 7 &= 9 \times 8 \times 125 - 9 \times 9 \times 8 \times 7 = (9 \times 8) \times (125 - 9 \times 7) = 72 \times (125 - 63) \\ &= 72 \times 62 = 62 \times 72 . \end{aligned}$$

So A is the correct option.

For further investigation

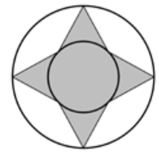
24.1 How many five-digit integers have at least one digit repeated?

24.2 Find a formula for the number of n-digit integers which have at least one digit repeated, where n is a positive integer with $2 \le n \le 10$.

24.3 If you pick a ten-digit integer at random, what is the probability that it has at least one digit repeated?

24.4 If you pick an eleven-digit integer at random, what is the probability that it has at least one digit repeated?

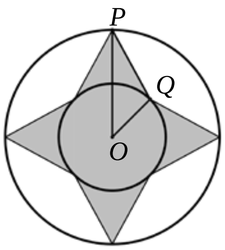
25. The diagram shows two concentric circles with radii of 1 and 2 units, together with a shaded octagon, all of whose sides are equal.



What is the length of the perimeter of the octagon?

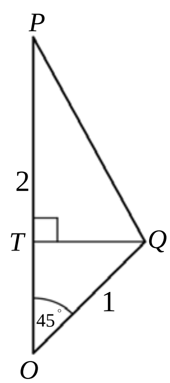
A $8\sqrt{2}$ B $8\sqrt{3}$ C $8\sqrt{3}\pi$ D $2\sqrt{5+2\sqrt{2}}$ E $8\sqrt{5-2\sqrt{2}}$

Solution: **E**



We need to find the length of one of the sides of the octagon. We consider the side PQ as shown in the diagram on the left. This is a side of the triangle PQO, where O is the centre of both circles. In this triangle OP has length 2, OQ has length 1 and $\angle POQ = 45^\circ$. For clarity we have drawn this triangle in a larger scale below.

We let T be the point where the perpendicular from Q *to* OP meets OP.



In the right-angled triangle QTO, $\angle TOQ = 45^\circ$. Therefore $\angle TQO = 45^\circ$ and so the triangle is isosceles. So $OT = TQ$. Since $OQ = 1$, we have, using Pythagoras' Theorem, that $OT^2 + OT^2 = 1$. Hence $OT^2 = \frac{1}{2}$ and therefore

$$OT = TQ = \frac{1}{\sqrt{2}}.$$

It follows that $PT = 2 - \frac{1}{\sqrt{2}}$. Therefore, by Pythagoras' Theorem applied to triangle PTQ,

$$PQ^2 = PT^2 + TQ^2 = \left(2 - \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \left(4 - 2\sqrt{2} + \frac{1}{2}\right) + \frac{1}{2} = 5 - 2\sqrt{2}.$$

It follows that $PQ = \sqrt{5-2\sqrt{2}}$ and hence the length of the perimeter of the octagon is $8\sqrt{5-2\sqrt{2}}$.

For further investigation

25.1 Use your knowledge of the lengths of the sides in triangle OTQ to find an exact expression for the value of $\cos 45^\circ$.

25.2 Another way to calculate the length of PQ is to apply the *Cosine Rule* to the triangle PQO.

(a) If you know the Cosine Rule, use it to verify that the length of PQ is $8\sqrt{5-2\sqrt{2}}$.

(b) If you haven't seen the Cosine Rule, or have forgotten it, find out what it says, and then use it to calculate the length of PQ.